Mathematics: analysis and approaches
Higher Level
Paper 2

Name

Date: ____________________

2 hours

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your name on each answer sheet and attach them to this examination paper.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

exam: 13 pages

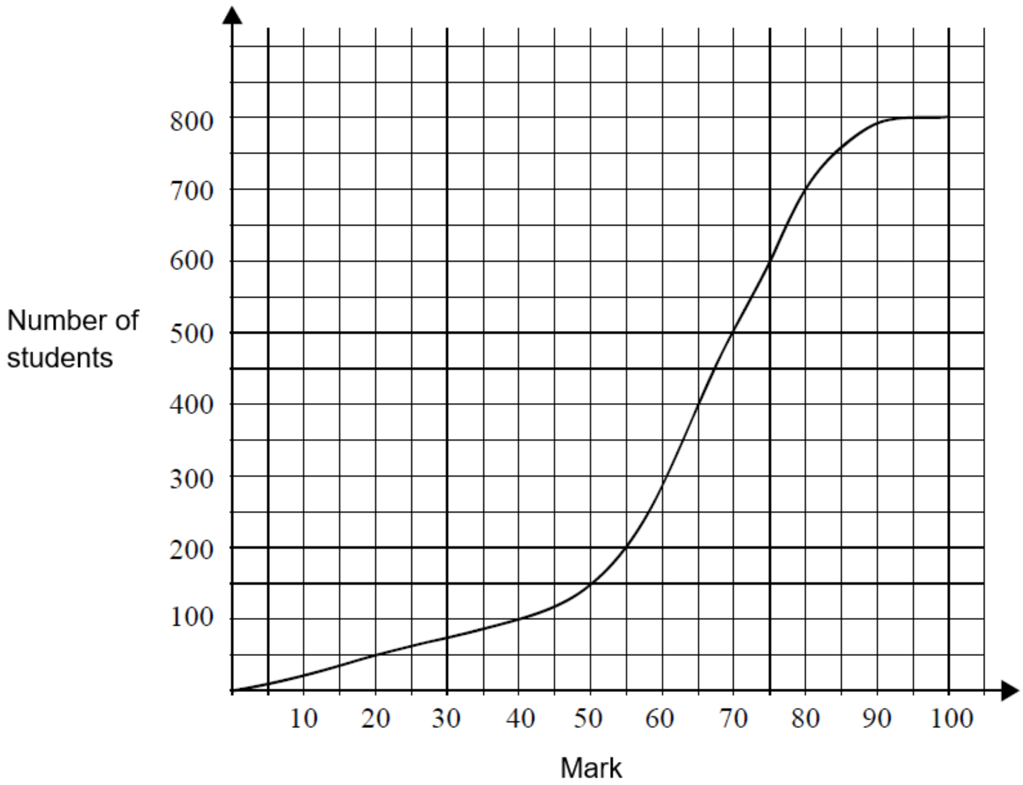
Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

A test marked out of 100 is written by 800 students. The cumulative frequency graph for the marks is shown below.



(a) Write down the number of students who scored 40 marks or less on the test. [2]

(b) The middle 50% of test results lie between marks a and b, where $a < b$. Find the value of a and the value of b. [3]

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2. [Maximum mark: 5]

A sum of $3000 is invested at a compound interest rate of 4.6% per year.

(a) Find the value of the investment at the end of seven years. [2]

(b) The value of the investment will exceed $5000 after x full years. Calculate the minimum value of x. [3]

3. [Maximum mark: 6]

(a) Show that $\dfrac{x-4}{2x^2-x-1}=\dfrac{3}{2x+1}-\dfrac{1}{x-1}$. [3]

(b) Hence, find $\displaystyle\int\frac{x-4}{2x^2-x-1}\,\mathrm{d}x$ [3]

4. [Maximum mark: 7]

Find the two possible lengths of AC in triangle ABC given that angle A is $42°$, $\mathrm{AB} = 12.7$ cm and $\mathrm{BC} = 10.2$ cm.

5. [Maximum mark: 6]

The probability of obtaining heads on a biased coin is $\frac{3}{5}$. The coin is tossed 500 times.

(a) (i) Write down the mean number of heads. [1]

(ii) Find the standard deviation of the number of heads. [2]

(b) Find the probability that the number of heads obtained is less than one standard deviation away from the mean. [3]

6. [Maximum mark: 7]

For the students at a certain secondary school, it is determined that the time it takes to travel to school is normally distributed with mean μ and standard deviation σ. It is found that 4% of students take less than 5 minutes to get to school and 70% take less than 25 minutes. Find the value of μ and the value of σ.

7. [Maximum mark: 6]

(a) Expand $\dfrac{1+x}{(1-4x)^3}$ in ascending powers of x up to and including the term in x^3. [4]

(b) Determine the values of x for which the expansion is valid. [2]

8. [Maximum mark: 7]

The acceleration, in m s^{-2}, of a particle at time t seconds is given by the function

$$a(t)=\frac{3}{t}+2t\sin 2t,\ \ t\geq 1$$

Given that the particle is at rest when $t=1$, find the velocity of the particle when $t=6$. [7]

9. [Maximum mark: 7]

If α and β are the roots of the equation $px^2+qx+q=0$, where $p,q\in\mathbb{Z}$, find the equation with Integer coefficients whose roots are $\dfrac{1}{\alpha+1}$ and $\dfrac{1}{\beta+1}$.

Do **not** write solutions on this page.

Section B

Answer **all** the questions on the answer sheets provided. Please start each question on a new page.

10. [Maximum mark: 16]

Each day, a factory records the number (x) of boxes it produces and the total production costs (y) in dollars. The results for nine days are shown in the following table.

boxes (x)	26	44	65	43	50	31	68	46	57
costs (y)	400	582	784	625	699	448	870	537	724

(a) Let L_1 be the regression line of y on x that can be written in the form $y = mx + c$. Write down the equation for L_1. [2]

(b) Interpret the meaning of

(i) the gradient m; [1]

(ii) the y-intercept c; [1]

(c) Estimate the cost of producing 60 boxes. [3]

(d) The factory sells the boxes for $19.99 each. Find the least number of boxes that the factory should produce in one day in order to make a profit. [3]

(e) Comment on the appropriateness of using the regression line L_1 to estimate the cost of producing 1000 boxes. [2]

(f) Let L_2 be the regression line of x on y that can be written in the form $x = ay + b$. Write down the equation for L_2. [2]

(g) Estimate the number of boxes produced when the total production cost is $550. [2]

Do **not** write solutions on this page.

11. [Maximum mark: 17]

(a) Consider the function h defined by $h(x)=\dfrac{e^x}{\sqrt{x}}$, for $0<x\leq 3$.

(i) Sketch the graph of h. [2]

(ii) Find $h'(x)$. [3]

(iii) Write down an expression, in terms of x. representing the gradient of the normal to the curve at any point. [2]

(b) Let P be the point (x,y) on the graph of h, and Q the point $(1,0)$.

(i) Find the gradient of (PQ) in terms of x. [2]

(ii) Given that the line (PQ) is a normal to the graph of h at the point P, find the minimum distance from point Q to the graph of h. [5]

(c) Consider the function g defined by $g(x)=\dfrac{e^x}{c\sqrt{x}}$, for $0<x\leq 3$, where $c\in\mathbb{R}$.

Let R be the point on the x-axis that is nearest to the graph of g. Determine the exact coordinates of R. [3]

Exam continues on the next page.

Do **not** write solutions on this page.

12. [Maximum mark: 21]

Newton's law of cooling states that the rate at which an object's temperature, T, decreases is proportional to the difference between the temperature of the object and the temperature of its surroundings – also known as the ambient temperature, A. This law can be expressed as the differential equation $\frac{\mathrm{dT}}{\mathrm{d}t} = -k(\mathrm{T} - \mathrm{A})$ where k is a constant and $\mathrm{T} - \mathrm{A} > 0$.

(a) State whether k is always positive or always negative and provide a reason. [2]

At $t = 0$ minutes, a hard-boiled egg has a temperature of $98°$C and is placed in a pan under running water such that the water has a constant temperature of $18°$C. At $t = 5$ minutes, the egg has a temperature of $38°$C.

(b) (i) Show that the temperature of the hard-boiled egg, T (in $°\mathrm{C}$), can be modelled by the function $\mathrm{T}(t) = 80\mathrm{e}^{-kt} + 18$ where t is time (in minutes). [4]

(ii) Find how many minutes it takes for the hard-boiled egg to cool to $20°$C from when the egg is placed in the pan. [4]

It is determined that the temperature of the water, A, is not constant and decreases slowly according to the function $\mathrm{A}(t)$. Hence, the rate of change of the temperature of the egg is given by the differential equation $\frac{\mathrm{dT}}{\mathrm{d}t} = -k(\mathrm{T} - \mathrm{A}(t))$.

(c) (i) Given that $\mathrm{A}(t) = \mathrm{A}_0\mathrm{e}^{-\lambda t}$ where $\mathrm{A}_0 = 18°$C (temperature of the water at $t = 0$), $k = 0.25$, and $\lambda = 0.2$, show that the rate of change of the temperature of the egg can be modelled by the differential equation $\frac{\mathrm{dT}}{\mathrm{d}t} = -0.25\mathrm{T} + 4.5\mathrm{e}^{-0.2t}$. [2]

(ii) By solving this differential equation, determine a function for T in the form $\mathrm{T}(t) = a\,\mathrm{e}^{-0.2t} + \mathrm{b}\,\mathrm{e}^{-0.25t}$, where $a, b \in \mathbb{R}$. [7]

(iii) Hence, find how many minutes it takes for the hard-boiled egg to cool to $20°$C from when the egg is placed in the pan. [2]